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1983 J. Phys. A: Math. Gen. 16 861

(http://iopscience.iop.org/0305-4470/16/4/022)

J. Phys. A: Math. Gen. 16 (1983) 861–874. Printed in Great Britain

# Supermanifold description of the BRS symmetries of skewsymmetric tensor gauge fields

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Received 7 May 1982, in final form 16 July 1982

**Abstract.** We extend Tulczyjew's geometrical formulation of skewsymmetric tensor gauge fields as connections on generalised principal fibre bundles to the category of supermanifolds. Given a smooth $d$-dimensional manifold $M$ and a $k$-form field on it, we construct a suitable supersmooth generalised principal fibre bundle $\mathscr{P}$ over a $(d, 2)$-dimensional supermanifold $\mathscr{M}$ such that the BRS symmetries of the theory have a natural geometrical interpretation.

## 1. Introduction

The theory of electromagnetism can be generalised in two different directions. On the one hand, one can go from an abelian to a non-abelian gauge group, maintaining the tensorial character of the field and thus obtaining the Yang–Mills theories. On the other hand, it is possible to go from the vector potential one-form to a $(k+1)$-form retaining the abelian character of the gauge group. The latter kind of field is receiving increased attention in several branches of theoretical physics: such fields are now involved in the U(1) problem and the confinement problem in QCD and they appear in supergravity (see e.g. Townsend 1981).

This paper is a contribution to a program initiated by Bonora and Tonin (1981) and carried on in a series of papers by Bonora, Pasti, Tonin and Marchetti (Bonora *et al* 1981a, b, c, Marchetti and Tonin 1981) aimed at giving a simple geometrical way of deriving all the auxiliary terms in the Lagrangian which are needed for the quantisation of gauge fields. This is possible for theories which admit two Becchi–Rouet–Stora symmetries (which we refer to as BRS and anti-BRS or $\overline{\text{BRS}}$ (Curci and Ferrari 1976)), i.e. Yang–Mills and the theory of skewsymmetric gauge fields (differential forms). Here we give a mathematically precise formulation, in terms of supermanifolds and (super) generalised principal fibre bundles, of the construction of the Lagrangian for $(k+1)$-forms (Marchetti and Tonin 1981) which made use of the language of superfields.

Let us sketch the general philosophy of our construction. Let $M$ be a $d$-dimensional real manifold, $\Lambda^k(M)$ the bundle of $k$-forms over $M$ and $E_k(M)$ the $C^\infty(M)$-module of $C^\infty$-sections of $\Lambda^k(M)$; we consider a field $A \in E_{k+1}(M)$ whose dynamics is given by

$$S_{\text{inv}} = \int_M \mathscr{L}_{\text{inv}} = -\tfrac{1}{2}\int_M F \wedge {}^*F \tag{1.1}$$

 

where $F = \mathrm{d}A$. Invariance here refers to the well known 'gauge' transformation

$$A \to A + \mathrm{d}\Lambda \qquad \Lambda \in E_k(M). \tag{1.2}$$

For $k = 0$ this is Maxwell's theory of electromagnetism, but for $k \neq 0$ it cannot be interpreted geometrically as the theory of a connection in a principal fibre bundle, as the gauge symmetry (1.2) would suggest. Tulczyjew (1979) has worked out a geometrical framework for the description of such theories, which he calls generalised principal fibre bundles. Roughly speaking, a $(k+1)$-form $A$ can be regarded as a connection in a bundle $P$ which is locally isomorphic to $\Lambda^k(M)$ (in § 2 we will briefly repeat this construction over a general supermanifold $\mathcal{M}$). For the sake of quantisation we need to supplement (1.1) with a number of gauge-fixing and ghost terms in such a way that the total Lagrangian exhibits BRS and $\overline{\text{BRS}}$ symmetries. In order to embody these properties into the geometry, we go over to a suitable $(d, 2)$-dimensional supermanifold $\mathcal{M}$ and generalised principal fibre bundle $\mathcal{P}$ over $\mathcal{M}$; the $(k+1)$-form $A$ describing a connection in $P$ is carried over to a $(k+1)$-(even) form $\mathcal{A}$ over $\mathcal{M}$ describing a connection in $\mathcal{P}$, such that its curvature vanishes along the two new odd dimensions (see (4.1)). This allows us to identify the generators of translations along these odd dimensions as the BRS and $\overline{\text{BRS}}$ charges (Bonora and Tonin 1981a). All the auxiliary terms which are needed for quantisation can then be obtained in the Feynman gauge upon performing the superfield expansion of $\mathcal{A}$ in the expression

$$\tfrac{1}{2}\mathrm{i} \int_{\mathcal{M}} \mathcal{A} \wedge {}^*\mathcal{A} \tag{1.3}$$

and integrating over the odd variables.

The Hodge operator used here was defined by Berezin (1979) (see also Napolitano and Sciuto (1981)). It makes use of a Riemannian structure which in the case of interest to us, when $M = GM \times Q_1 \times Q_1$, is given by $Gg + 2\mathrm{i}\, \mathrm{d}\theta \otimes \mathrm{d}\bar{\theta}$, $g$ being a Riemannian structure in $M$, $\theta$ and $\bar{\theta}$ coordinates in the odd dimensions and $G$ the Grassmann enlargement (see below, point (d)).

We conclude this introduction by establishing what we mean by supermanifold, using the language of category theory (e.g. Lang 1972). There are at present several definitions of supermanifold being used in the mathematical and physical literature; some of them turn out to be equivalent to others, and some not. Since the subject seems to be still somewhat controversial, we will not adopt one particular definition, but rather require that the definition to be used satisfy some general properties, which we now list.

(a) Given a fixed graded Grassmann algebra $Q = Q_0 \otimes Q_1$, there exists a category of smooth $Q$-supermanifolds, analogous to the category of smooth real manifolds.

(b) In the same way as one defines smooth vector bundles using manifolds, it is possible to define smooth super vector bundles using supermanifolds; in the same way that one defines functors $T$, $T^*$, $\Lambda^k$ $(k \geq 1)$ from the category of real manifolds to the category of real vector bundles which map a manifold to its tangent, cotangent and exterior bundles, it is possible to define analogous functors (still denoted $T$, $T^*$, $\Lambda^k$) for supermanifolds.

(c) There exists a surjective functor $B$ from the category of smooth $Q$-supermanifolds to the category of smooth real manifolds which associates to each $(m, n)$-dimensional ($m$ 'even' and $n$ 'odd' dimensional) $Q$-supermanifold an $m$-dimensional real manifold, to be called its 'body'.

(d) There exists an injective functor $G$ from the category of smooth real manifolds to the category of smooth $Q$-supermanifolds which gives, in a sense, the 'simplest' $Q$-supermanifold having the original manifold as its body; we will refer to this as the 'Grassmann enlargement'. If the manifold is $m$-dimensional, its Grassmann enlargement will be $(m, 0)$-dimensional.

(e) There exists a forgetful functor $F$ from the category of smooth $Q$-supermanifolds to the category of smooth real manifolds (eventually Banach manifolds, if $Q$ is infinite-dimensional, as we will assume); if $\dim Q = 2p$, an $(m, n)$-dimensional supermanifold is mapped to a $p(m+n)$-dimensional real manifold.

(f) The functors $B$, $G$ commute with the functors $T$, $T^*$, $\Lambda^k$.

For definiteness one may bear in mind the definition of Rogers' supermanifold (Rogers 1980, Jadczyk and Pilch 1981); it is not known, however, how large the domain of definition of the functor is in this category. This problem will be studied in a separate paper, where we will also prove point (f) in this context (Marchetti and Percacci 1981); for the time being, we will call $G^\infty$-Man the subcategory of Rogers manifolds which admit a body (a Rogers supermanifold $\mathcal{M}$ with atlas $\{(\mathcal{U}_A, \psi_A)\}$ has a body $\hat{M}$ with atlas $\{(\hat{U}_A, \hat{\Psi}_A)\}$ if there exists a unique surjective map $\varphi: \mathcal{M} \to \hat{M}$ such that $\hat{U}_A = \varphi(\mathcal{U}_A)$ and $\varepsilon \circ \psi_A = \Psi_A \circ \varphi$; if $\mathscr{f}: \mathcal{M} \to \mathcal{N}$, $B\mathscr{f}: B\mathcal{M} \to B\mathcal{N}$ maps $\varphi(x) \mapsto \varphi(\mathscr{f}(x))$). The tangent bundle which we refer to in (b) is to be meant as the even tangent bundle defined by Jadczyk and Pilch (1981); the same holds for $T^*$ and $\Lambda^k$, e.g. $T^*\mathcal{M} = \mathrm{Mor}(T\mathcal{M}, \mathcal{M} \times Q_0)$, in the category of $G^\infty$-vector bundles. The Grassmann enlargement† has been defined in this context in Bonora *et al* (1981c). It should be mentioned that the Madrid group has given a new definition of supermanifold by formalising properties (a), (c), (e) (Hoyos *et al* 1981). Although property (e) will not be used explicitly in this paper, it is shared by several definitions of supermanifold (e.g. it holds for the Rogers supermanifold), and it seems at least desirable for physical applications. The reason for this is the following: in order to avoid well known nilpotency problems for the ghosts, one has to assume that the Grassmann algebra $Q$ has to be infinite dimensional; thus applying the functor $F$ we obtain an infinite-dimensional Banach manifold, and it is known that a proper geometrisation of the ghost fields does require infinite-dimensional manifolds (Leinaas and Olaussen 1981).

We conclude this section with some remarks concerning our notations.

Categories:

| | |
|---|---|
| $C^\infty$-Man | smooth real manifolds |
| $G^\infty$-Man | smooth Rogers $Q$-supermanifolds with body ($Q$ fixed) |
| $C^\infty$-VB | smooth real vector bundles |
| $G^\infty$-VB | smooth super vector bundles |
| $C^\infty$-GPB$^k(M)$ | smooth generalised principal fibre bundles over $M$, type $k$ |
| $G^\infty$-GPB$^k(\mathcal{M})$ | smooth generalised principal fibre bundles over $\mathcal{M}$, type $k$. |

Functors:

| | |
|---|---|
| $B: G^\infty\text{-Man} \to C^\infty\text{-Man}$ | body |
| $G: C^\infty\text{-Man} \to G^\infty\text{-Man}$ | Grassmann enlargement |
| $T: C^\infty\text{-Man} \to C^\infty\text{-VB}$ | tangent |
| $T: G^\infty\text{-Man} \to G^\infty\text{-VB}$ | tangent. |

We use throughout script characters for objects and morphisms in the $G^\infty$-categories, and Roman characters for objects and morphisms in the $C^\infty$-categories. A particular

† In fact, for the Grassmann enlargement also there are problems when $Q$ is infinite dimensional but these problems do not arise in the case of $G^\omega$ supermanifolds and so we will not worry about this here.

object or morphism in the $C^\infty$-category arising from the application of the functor $B$ will sometimes be denoted by a hat (e.g. $B\mathcal{M} = \hat{M}$, $B\not{f} = \hat{f}$), and similarly objects or morphisms in the $G^\infty$-category arising from the application of the functor $G$ will sometimes be denoted by a bar (e.g. $GM = \bar{\mathcal{M}}$, $Gf = \bar{\not{f}}$).

The functors $B$ and $G$ act on vector bundles in the following way: if

$$(\mathscr{E}, \pi, \mathcal{M}) \in G^\infty\text{-VB}, \qquad B(\mathscr{E}, \pi, \mathcal{M}) = (B\mathscr{E}, B\pi, B\mathcal{M})$$

and if

$$(E, \pi, M) \in C^\infty\text{-VB}, \qquad G(E, \pi, M) = (GE, G\pi, GM).$$

## 2. Generalised principal fibre bundles on a supermanifold

Acting with $\Lambda^k$ on a supermanifold $\mathcal{M}$, we obtain the $G^\infty$-vector bundles of $k$-forms $\Lambda^k(\mathcal{M}) = (\mathscr{P}^k, \pi^k, \mathcal{M})$, $k = 0, 1, \ldots$. Let $G^\infty(\mathcal{M}) = G_0^\infty(\mathcal{M}) \oplus G_1^\infty(\mathcal{M})$ denote the graded commutative algebra (which is also a graded $Q$-module) of $G^\infty$ functions from $\mathcal{M}$ to $Q$. $G_0^\infty(\mathcal{M})$ is the commutative algebra (which is also a $Q_0$-module) of $G^\infty$ functions from $\mathcal{M}$ to $Q_0$. $G_0^\infty(\mathcal{M})$ can be identified through the graph relation with the set of all sections of $\Lambda^0(\mathcal{M})$, which we denote $E_0(\mathcal{M})$; the set of all sections of $\Lambda^k(\mathcal{M})$ for $k > 0$ is a $G_0^\infty(\mathcal{M})$-module denoted $E_k(\mathcal{M})$. The group composition law in this module is the pointwise addition of forms; with respect to this, $E_k(\mathcal{M})$ are infinite-dimensional abelian Lie groups, and as such they possess infinite-dimensional commutative Lie algebras over $\mathbb{R}$, which we denote $\mathscr{E}_k(\mathcal{M})$. The distinction between $E_k$ and $\mathscr{E}_k$ is however purely formal, since an element $s$ of $E_k(\mathcal{M})$ is generated by an element $\sigma$ of $\mathscr{E}_k(\mathcal{M})$ with $s = \sigma$. Tulczyjew (1979) defines a group action $\gamma^k : \mathscr{P}^k \times E_k(\mathcal{M}) \to \mathscr{P}^k$ by

$$\gamma^k(p, s) = p + s(\pi^k(p)). \tag{2.1}$$

This action is effective but not free. We also define $\gamma_s^k : \mathscr{P}^k \to \mathscr{P}^k$ by $\gamma_s^k(p) = \gamma^k(p, s)$. Let now $c: I \to E_k(\mathcal{M})$ be a $C^1$ curve with $c(0) = s = \sigma$ and $\gamma^k_{c(t)}$ be the flow induced on $\mathscr{P}^k$; we define the fundamental vector field $W^k(\sigma)$ on $\mathscr{P}^k$ to be

$$W^k(\sigma)|_p = \frac{\mathrm{d}}{\mathrm{d}t}\gamma^k(p, c(t))|_{t=0}. \tag{2.2}$$

Notice that $W^k(\sigma)$ belongs to the $G_0^\infty(\mathscr{P}^k)$-module $\mathscr{T}(\mathscr{P}^k)$ of $G^\infty$-sections of $T\mathscr{P}^k$. Equation (2.2) defines a homomorphism $W^k : \mathscr{E}^k(\mathcal{M}) \to \mathscr{T}(\mathscr{P}^k)$. We now define the canonical $k$-form $\vartheta^k \in E_k(\mathscr{P}^k)$ by

$$(\vartheta^k | v_1, \ldots, v_k)|_p = (p | T\pi^k(v_1), \ldots, T\pi^k(v_k))|_{\pi(p)} \tag{2.3}$$

where $v_i \in T_p\mathscr{P}^k$ ($i = 1, \ldots, k$) for $k > 0$, and $\vartheta^k = pr_2$ for $k = 0$ (the canonical projection $\Lambda^0\mathcal{M} = \mathcal{M} \times Q_0 \to Q_0$) and the canonical $(k+1)$-form $\omega^k \in E_{k+1}(\mathscr{P}^k)$ by

$$\omega^k = \mathrm{d}\vartheta^k. \tag{2.4}$$

We summarise here the properties of the canonical forms, which generalise straightforwardly from the $C^\infty$ to the $G^\infty$ case:

$$i_{W^k(\sigma)}\vartheta^k = 0 \tag{2.5}$$

$$\mathscr{L}_{W^k(\sigma)}\vartheta^k = \pi^{k*}\sigma \tag{2.6}$$

$$\gamma_s^{k*}\vartheta^k = \vartheta^k + \pi^{k*}s \tag{2.7}$$

$$i_{W^k(\sigma)}\omega^k = \pi^{k*}\sigma \tag{2.8}$$

$$\mathscr{L}_{W^k(\sigma)}\omega^k = \pi^{k*}\,\mathrm{d}\sigma \tag{2.9}$$

$$\gamma_s^{k*}\omega^k = \omega^k + \pi^{k*}\,\mathrm{d}s. \tag{2.10}$$

If $\{y^a\}$ are local coordinates on $\mathscr{M}$ and $\{y^a, p_{a_1\ldots a_k}\}$ are local coordinates in $\mathscr{P}^k$, then locally we have

$$\vartheta^k|_{(y^a, p_{a_1\ldots a_k})} = (1/k!)p_{a_1\ldots a_k}\,\mathrm{d}y^{a_1}\wedge\ldots\wedge\mathrm{d}y^{a_k}.$$

Next we define a generalised principal fibre bundle over $\mathscr{M}$ modelled on $\Lambda^k\mathscr{M}$ to be a triple $(\mathscr{P}, \pi, \mathscr{M})$ where $\mathscr{P}$ is a $G^\infty$-manifold with a group action $\gamma\colon \mathscr{P}\times E_k(\mathscr{M})\to\mathscr{P}$ such that (i) $\mathscr{M}$ is the factor of $\mathscr{P}$ by the equivalence relation induced by $\gamma$, (ii) $\forall x\in\mathscr{M}$ $\exists$ a neighbourhood $\mathscr{U}$ of $x$ and an $\mathscr{M}$-isomorphism $\varphi\colon\pi^{-1}(\mathscr{U})\to(\pi^k)^{-1}(\mathscr{U})$ such that $\varphi(\gamma(p,s)) = \gamma^k(\varphi(p), s)$ $\forall s\in E_k(\mathscr{M})$. In analogy to (2.2) we can define the fundamental vector field $W(\sigma)$ generated by $\sigma$ on $\mathscr{P}$ and locally

$$W(\sigma) = T\varphi^{-1}(W^k(\sigma))$$

and $W$ may be regarded as a homomorphism $\mathscr{E}_k(\mathscr{M})\to\mathscr{T}(\mathscr{P})$. $A$ $(k+1)$-form $\alpha\in E_{k+1}(\mathscr{P})$ will be called a connection form if it satisfies the following properties:

$$i_{W(\sigma)}\alpha = \pi^*\sigma \qquad \sigma\in\mathscr{E}_k(\mathscr{M}) \tag{2.11}$$

$$\mathscr{L}_{W(\sigma)}\alpha = 0 \qquad \sigma = \mathrm{d}\tau \qquad \tau\in\mathscr{E}_{k-1}(\mathscr{M}). \tag{2.12}$$

The $(k+2)$-form $\beta = \mathrm{d}\alpha$ will be called curvature and satisfies

$$i_{W(\sigma)}\beta = 0 \tag{2.13}$$

$$\mathscr{L}_{W(\sigma)}\beta = 0. \tag{2.14}$$

One may check that

$$\gamma_s^*\alpha = \alpha + \pi^*\,\mathrm{d}s \tag{2.15}$$

$$\gamma_s^*\beta = \beta. \tag{2.16}$$

Let $\varphi$ be a trivialisation of $\mathscr{P}|_{\mathscr{U}}$; from (2.8), (2.9), (2.11), (2.12) there follows

$$i_{W(\sigma)}(\alpha - \varphi^*\omega^k) = 0 \tag{2.17}$$

$$\mathscr{L}_{W(\sigma)}(\alpha - \varphi^*\omega^k) = 0 \tag{2.18}$$

and therefore there exists $\mathscr{A}\in E_{k+1}(u)$ such that

$$\alpha - \varphi^*\omega^k = \pi^*\mathscr{A}. \tag{2.19}$$

Similarly from (2.13), (2.14) there follows

$$\beta = \pi^*\mathscr{F} \tag{2.20}$$

where

$$\mathscr{F} = \mathrm{d}\mathscr{A}. \tag{2.21}$$

If $\varphi'$ is a different trivialisation of $\mathscr{P}|_{\mathscr{U}}$, then

$$\varphi^*\omega^k - \varphi'^*\omega^k = \mathrm{d}(\varphi^*\vartheta^k - \varphi'^*\vartheta^k)$$

but again

$$i_{W(\sigma)}(\varphi^*\vartheta^k - \varphi'^*\vartheta^k) = 0 \qquad \mathscr{L}_{W(\sigma)}(\varphi^*\vartheta^k - \varphi'^*\vartheta^k) = 0$$

i.e. $\exists s \in E_k(\mathscr{U})$ such that

$$\varphi^*\vartheta^k - \varphi'^*\vartheta^k = -\pi^* s.$$

Therefore $\mathscr{A}$ is transformed into

$$\mathscr{A}' = \mathscr{A} + \mathrm{d}s. \tag{2.22}$$

This transformation can be considered as a gauge transformation for the field $\mathscr{A}$ on $\mathscr{M}$ which locally looks like

$$\mathscr{A}'_{a_1\ldots a_k}(y)\,\mathrm{d}y^{a_1}\wedge\ldots\wedge\mathrm{d}y^{a_k} = (\mathscr{A}_{a_1\ldots a_k}(y) + \partial_{a_1}s_{a_2\ldots a_k}(y))\,\mathrm{d}y^{a_1}\wedge\ldots\wedge\mathrm{d}y^{a_k}.$$

Let $G^\infty$-VB($\mathscr{M}$) ($C^\infty$-VB($M$)) denote the subcategory of $G^\infty$-VB ($C^\infty$-VB) of vector bundles over a fixed supermanifold $\mathscr{M}$ (manifold $M$). By assumption (f) of § 1, the functor $B$ maps the sequence of vector bundles $\Lambda^0\mathscr{M}, \Lambda^1\mathscr{M}, \Lambda^2\mathscr{M}, \ldots \in G^\infty$-VB($\mathscr{M}$) onto the sequence of vector bundles $\Lambda^0\hat{M}, \Lambda^1\hat{M}, \Lambda^2\hat{M}, \ldots \in C^\infty$-VB($\hat{M}$) where $\hat{M} = B\mathscr{M}$, and similarly the functor $G$ maps the sequence of vector bundles $\Lambda^0 M, \Lambda^1 M, \Lambda^2 M, \ldots \in C^\infty$-VB($M$) into the sequence of vector bundles $\Lambda^0\bar{\mathscr{M}}, \Lambda^1\bar{\mathscr{M}}, \Lambda^2\bar{\mathscr{M}}, \ldots \in G^\infty$-VB($\bar{\mathscr{M}}$) where $\bar{\mathscr{M}} = GM$. In the rest of this section we will prove that $B$ and $G$ act in a similar way on generalised principal fibre bundles and that the action can also be extended to the connections. We start with the following proposition.

*Proposition 2.23.* The functor $B$ maps the canonical forms $\vartheta^k \in E_k(\mathscr{P}^k)$ and $\omega^k \in E_{k+1}(\mathscr{P}^k)$ into the canonical forms $\Theta^k \in E_k(P^k)$ and $\Omega^k \in E_{k+1}(P^k)$ where $\mathscr{P}^k$ is the total space of $\Lambda^k\mathscr{M}$ and $P^k$ is the total space of $\Lambda^k\hat{M}$.

*Remark.* Here $\vartheta^k$ has to be regarded as a morphism of $G^\infty$-manifolds $\mathscr{P}^k \to \Lambda^k\mathscr{P}^k$ and hence it makes sense to write $B\vartheta^k$; the same holds for the other forms.

*Lemma 2.24.* If $\gamma \in E_k(\mathscr{M})$ and $v \in \mathscr{T}(\mathscr{M})$, $B(i_v\gamma) = i_{Bv}B\gamma$.

*Proof.* Let $p \in \mathscr{M}$; then $B\gamma: \varphi(p) \mapsto \varphi(\gamma_p)$, $Bv: \varphi(p) \mapsto \varphi(v_p)$ and $i_{Bv}B\gamma: \varphi(p) \mapsto i_{\varphi(v_p)}\varphi(\gamma_p)$. But $i: \Lambda^k\mathscr{M} \times T\mathscr{M} \to \Lambda^{k-1}\mathscr{M}$ is a morphism of $G^\infty$-Man and hence $i_{\varphi(v_p)}\varphi(\gamma_p) = \varphi(i_{v_p}\gamma_p)$ which is the image of $p$ under $B(i_v\gamma)$.

*Lemma 2.25.* If $\gamma \in E_k(\mathscr{M})$, $v \in \mathscr{T}(\mathscr{M})$, (i) $B\,\mathrm{d}\gamma = \mathrm{d}B\gamma \in E_{k-1}(\hat{M})$ and (ii) $B\mathscr{L}_v\gamma = \mathscr{L}_{Bv}B\gamma \in E_k(\hat{M})$.

*Proof.* It is easiest to prove this in local coordinates. If $\psi$ is a local coordinate system on a neighbourhood $\mathscr{U} \subset \mathscr{M}$, and $\mathscr{M}$ is $(m, n)$ dimensional, let $y^a$, $a = 1, \ldots, m+n$, denote the coordinate functions on $\mathscr{U}$: $y^a = pr_a \circ \psi$ where $pr_a$ denotes the projection on the $a$th factor of $Q^{m,n} = Q_0^m \oplus Q_1^n$. Now on functions ($k = 0$) $\mathrm{d}f = Tf$ and thus by assumption (f) $B\,\mathrm{d}f = \mathrm{d}Bf$. In particular, on the coordinate functions $B\,\mathrm{d}y^a = \mathrm{d}By^a = \mathrm{d}(\varepsilon \circ y^a)$; let us make the convention that $a = \mu$, $a = 1, \ldots, m$, label the even coordinates and $a = m + \alpha$, $\alpha = 1, \ldots, n$, label the odd coordinates. Then

$$B\,\mathrm{d}y^\mu = \mathrm{d}\hat{y}^\mu \qquad B\,\mathrm{d}y^\alpha = 0$$

where we have called $\hat{y}^{\mu} = By^{\mu}$ the real coordinates of $Bu$ defined by (2.6). Then

$$\begin{aligned} B\,\mathrm{d}\gamma &= B((1/k!)\,\mathrm{d}\gamma_{a_1\ldots a_k} \wedge \mathrm{d}y^{a_1} \wedge \ldots \wedge \mathrm{d}y^{a_k}) \\ &= (1/k!)\,\mathrm{d}B\gamma_{\mu_1\ldots\mu_k} \wedge \mathrm{d}\hat{y}^{\mu_1} \wedge \ldots \wedge \mathrm{d}\hat{y}^{\mu_k} = \mathrm{d}B\gamma. \end{aligned}$$

Statement (ii) is a direct consequence of the formula $\mathscr{L}_v = \mathrm{d} \circ i_v + i_v \circ \mathrm{d}$ on $E_k(\mathcal{M})$ and lemma 2.24.

*Proof of proposition.* In (2.3) $p$ appears on the left as a point of $\mathscr{P}^k$ and on the right as a $k$-form on $\mathcal{M}$ at $\pi^k(p)$. Thus the application of $B$ to it can be expressed either with $\varphi(p) \in P^k = B\mathscr{P}^k$ or with $Bp: T\hat{M} \times \ldots \times T\hat{M} \to \mathbb{R}$, the meaning being obviously the same. Then applying $B$ on both sides of (2.3) and using lemma 2.24 gives

$$\begin{aligned} &(B\vartheta^k|Bv_1, \ldots, Bv_k)|_{\varphi(p)} \\ &\qquad = B(\vartheta^k|v_1, \ldots, v_k)|_{\varphi(p)} \\ &\qquad = B(p|T\pi^k(v_1), \ldots, T\pi^k(v_k))|_{\varphi(p)} \\ &\qquad = (Bp|T\hat{\pi}^k(Bv_1), \ldots, T\hat{\pi}^k(Bv_k))|_{\varphi(p)}. \end{aligned}$$

The statement for $\omega^k$ is a trivial consequence of lemma 2.25. □

It would be desirable to define the categories of generalised principal fibre bundles (over real or supermanifolds) but this is actually impossible. For let $\upsilon: \mathscr{P}_1 \to \mathscr{P}_2$ be a fibre preserving map: $\pi_2(\upsilon(p)) = f(\pi_1(p))$ for $p \in \mathscr{P}_1$ and some function $f: \mathcal{M}_1 \to \mathcal{M}_2$; then one should impose $\gamma_2(\upsilon(p), \lambda(s)) = \upsilon(\gamma_1(p, s))$, $\lambda$ being a homomorphism $E_k(\mathcal{M}_1) \to E_k(\mathcal{M}_2)$. But a natural homomorphism in general does not exist, because forms pull-back under the action of $f$. Consequently, we must content ourselves with a more restricted definition. We define a category $G^{\infty}$-GPB$^k(\mathcal{M})$ (resp $C^{\infty}$-GPB$^k(M)$) whose objects are the generalised principal fibre bundles over $\mathcal{M} \in G^{\infty}$-Man modelled on $\Lambda^k\mathcal{M}$ (resp the generalised principal fibre bundles over $M \in C^{\infty}$-Man modelled on $\Lambda^k M$) and whose morphisms are fibre maps $\upsilon: \mathscr{P}_1 \to \mathscr{P}_2$ such that, with obvious notation, $\gamma_s^{(2)} \circ \upsilon = \upsilon \circ \gamma_s^{(1)}$, $s \in E_k(\mathcal{M})$ (resp $s \in E_k(M)$).

We define the action of $B$ on $G^{\infty}$-GPB$^k(\mathcal{M})$ by

$$B(\mathscr{P}, \pi, \mathcal{M}) = (B\mathscr{P}, B\pi, B\mathcal{M}) =: (\hat{P}, \hat{\pi}, \hat{M}).$$

This is a generalised principal fibre bundle over $\hat{M} = B\mathcal{M}$ because if $\varphi$ is a local trivialisation of $\mathscr{P}$, $\hat{\varphi} = B\varphi$ is a local trivialisation of $\hat{P}$ satisfying condition (ii) of the definition. Furthermore, if $(\upsilon_1, f_1)$ and $(\upsilon_2, f_2)$ are morphisms $\mathscr{P}_1 \to \mathscr{P}_2$ and $\mathscr{P}_2 \to \mathscr{P}_3$, the following diagrams hold:

$$\downarrow = \varphi \qquad \begin{tikzcd} \mathscr{P}_1 \arrow[rr, "\upsilon_1"] \arrow[dr, "\pi_1"] \arrow[dd] & & \mathscr{P}_2 \arrow[rr, "\upsilon_2"] \arrow[dr, "\pi_2"] \arrow[dd] & & \mathscr{P}_3 \arrow[dr, "\pi_3"] \arrow[dd] & \\ & \mathcal{M} \arrow[rr, "f_1"] \arrow[dd] & & \mathcal{M} \arrow[rr, "2"] \arrow[dd] & & \mathcal{M} \arrow[dd] \\ \hat{P}_1 \arrow[rr, "B\upsilon_1"] \arrow[dr, "\hat{\pi}_1"] & & \hat{P}_2 \arrow[rr, "B\upsilon_2"] \arrow[dr, "\hat{\pi}_2"] & & \hat{P}_3 \arrow[dr, "\hat{\pi}_3"] & \\ & \hat{M} \arrow[rr, "Bf_1"] & & \hat{M} \arrow[rr, "Bf_2"] & & \hat{M} \end{tikzcd} \tag{2.26}$$

$$\hat{s} = Bs \in E_k(\hat{M}) \qquad \begin{tikzcd}
\mathscr{P}_1 \arrow[rr, "u_1"] \arrow[dr, "\gamma_s"] \arrow[dd] & & \mathscr{P}_2 \arrow[rr, "u_2"] \arrow[dr, "\gamma_s"] \arrow[dd] & & \mathscr{P}_3 \arrow[dr, "\gamma_s"] \arrow[dd] & \\
& \mathscr{P}_1 \arrow[rr, "u_1"] \arrow[dd] & & \mathscr{P}_2 \arrow[rr, "u_2"] \arrow[dd] & & \mathscr{P}_3 \arrow[dd] \\
\hat{P}_1 \arrow[rr, "Bu_1"] \arrow[dr, "B\gamma_{\hat{s}}"'] & & \hat{P}_2 \arrow[rr, "Bu_2"] \arrow[dr, "B\gamma_{\hat{s}}"'] & & \hat{P}_3 \arrow[dr, "B\gamma_{\hat{s}}"'] & \\
& \hat{P}_1 \arrow[rr, "Bu_1"] & & \hat{P}_2 \arrow[rr, "Bu_2"] & & \hat{P}_3
\end{tikzcd} \tag{2.27}$$

and hence

*Proposition 2.28.* $B$ is a surjective functor for the category $G^\infty$-GPB$^k(\mathscr{M})$ to the category $C^\infty$-GPB$^k(B\mathscr{M})$.

This result extends also to generalised principal fibre bundles with connection, in the following sense.

*Proposition 2.29.* If $(\mathscr{P}, \pi, \mathscr{M}) \in G^\infty$-GPB$^k(\mathscr{M})$ and $\alpha$ is a connection form in $\mathscr{P}$, $B\alpha$ is a connection form in $(B\mathscr{P}, B\pi, B\mathscr{M})$. If $\beta$ is the curvature form of $\alpha$, $B\beta$ is the curvature of $B\alpha$.

*Proof.* If $\sigma \in \mathscr{E}_k(\mathscr{M})$, $p \in \mathscr{P}$ and $c$ is as in (2.2), using (2.27)

$$BW(\sigma)|_{\varphi(p)} = B\left(\frac{\mathrm{d}}{\mathrm{d}t}\gamma(p, c(t))|_{t=0}\right) = \frac{\mathrm{d}}{\mathrm{d}t}B\gamma(\varphi(p), Bc(t)) = W(B\sigma)|_{\varphi(p)}$$

and thus $BW(\sigma) = W(B\sigma)$. Now applying $B$ on both sides of (2.11), (2.12) and using lemmas 2.24, 2.25 the result follows at once.

From propositions 2.23 and 2.29, it follows that in $\hat{M}$

$$B\alpha - \hat{\varphi}^*\Omega^k = \hat{\pi}^*A \tag{2.30}$$

$$B\beta = \hat{\pi}^*F \tag{2.31}$$

$$F = \mathrm{d}A \tag{2.32}$$

where $A = B\mathscr{A}$, $F = B\mathscr{F}$, and moreover a change of trivialisation (a 'super' gauge transformation) $\varphi \to \varphi'$ of $\mathscr{P}|_{\mathscr{U}}$ gives rise to a change of trivialisation (gauge transformation), $\hat{\varphi} \to \hat{\varphi}'$ of $\hat{P}|_{B\mathscr{U}}$ such that the transformation law for $A$ is

$$A' = A + \mathrm{d}Bs, \tag{2.33}$$

$s$ being the $k$-form appearing in (2.22). Thus the whole structure of the generalised principal fibre bundle over $\mathscr{M}$ projects upon a similar structure on $\hat{M}$.

We now turn to the functor $G$. The situation is completely analogous to that of $B$ and therefore we only state the results, without proofs.

*Lemma 2.34.* If $\gamma \in E_k(M)$ and $v \in \mathscr{T}(M)$

(i) $G(i_v\gamma) = i_{Gv}G\gamma$

(ii) $G\,\mathrm{d}\gamma = \mathrm{d}G\gamma$

(iii) $G(\mathscr{L}_v\gamma) = \mathscr{L}_{Gv}G\gamma$.

*Proposition 2.35.* The functor $G$ maps the canonical forms $\Theta^k \in E_k(P^k)$ and $\Omega^k \in E_{k+1}(P^k)$ into the canonical forms $\vartheta^k \in E_k(\bar{\mathscr{P}}^k)$ and $\omega^k \in E_{k+1}(\bar{\mathscr{P}}^k)$ where $P^k$ is the total space of $\Lambda^k M$ and $\bar{\mathscr{P}}^k$ is the total space of $\Lambda^k \bar{\mathscr{M}} = \Lambda^k GM$.

The action of $G$ can be defined on the category $C^\infty$-GPB$^k(M)$ by

$$G(P, \pi, M) = (GP, G\pi, GM) = (\bar{\mathscr{P}}, \bar{\pi}, \bar{\mathscr{M}})$$

and

*Proposition 2.36.* $G$ is a functor from the category $C^\infty$-GPB$^k(M)$ to the category $G^\infty$-GPB$^k(\bar{\mathscr{M}})$.

The result extends also to generalised principal fibre bundles with connection, in the following sense.

*Proposition 2.37.* If $(P, \pi, M) \in C^\infty$-GPB$^k(M)$ and $\alpha$ is a connection form in $P$, $G\alpha$ is a connection form in $(\bar{\mathscr{P}}, \bar{\pi}, \bar{\mathscr{M}})$. If $\beta$ is the curvature form of $\alpha$, $G\beta$ is the curvature form of $G\alpha$.

From propositions 2.35 and 2.37 it follows that in $\bar{\mathscr{P}}$

$$G\alpha - \bar{\varphi}^* \omega^k = \bar{\pi}^* \mathscr{A} \qquad (\bar{\varphi} = G\varphi) \tag{2.38}$$

$$G\beta = \bar{\pi}^* \mathscr{F} \tag{2.39}$$

$$\mathscr{F} = \mathrm{d}\mathscr{A} \tag{2.40}$$

where $\mathscr{A} = GA$, $\mathscr{F} = GF$, and moreover a change of trivialisation (a gauge transformation) $\varphi \to \varphi'$ of $P|_U$ gives rise to a change of trivialisation (a 'super' gauge transformation) $\bar{\varphi} \to \bar{\varphi}'$ on $\bar{\mathscr{P}}|_{GU}$ such that the transformation law for $\mathscr{A}$ is

$$\mathscr{A}' = \mathscr{A} + \mathrm{d}Gs \tag{2.41}$$

if $A' = A + \mathrm{d}s$. Thus the whole structure of the generalised principal fibre bundle over $M$ can be lifted to a similar structure on $\bar{\mathscr{M}}$.

## 3. Geometrical meaning of the BRS transformations

In the previous section we have seen how we can describe the gauge symmetry of $(k+1)$-forms on a supermanifold and how this is related to the gauge symmetry on the body manifold. It will be apparent in the following that the simple Grassmann enlargement of a gauge structure on $M$ (an element of $C^\infty$-GPB$^k(M)$ with connection) does not carry any information beyond the one already contained in its body, and hence is not enough to embody the BRS structure of the quantum theory; on the other hand, a general gauge structure on a supermanifold $\mathscr{M}$ such that $M = B\mathscr{M}$ (an element of $G^\infty$-GPB$^k(\mathscr{M})$ with connection) may not exhibit the desired symmetries; thus the problem is to construct a suitable gauge structure on a suitable $\mathscr{M}$ which enjoys them. This is the content of the present section. The construction will be based on previous work by Marchetti and Tonin (1981), where it was shown using superfield methods how to obtain the total Lagrangian and its BRS symmetries. Stated in the present language, this goes roughly as follows: from the $d$-dimensional real manifold $M$ one constructs a $(d, 2)$-dimensional supermanifold $\mathcal{M} = GM \times Q^{0,2}$ with projections $q_0 : \mathcal{M} \to GM$, $q_1 : \mathcal{M} \to Q^{0,2}$ and given a field $A \in E_{k+1}(M)$ one constructs a field $\mathcal{A} \in E_{k+1}(\mathcal{M})$ such that $B\mathcal{A} = A$ and

$$\mathcal{F} = q_0^* GF \tag{3.1}$$

which is the condition that allows us to identify the BRS and $\overline{\text{BRS}}$ transformations as translations along the two odd coordinates of $\mathcal{M}$. This is equivalent to

$$\mathcal{A} = q_0^* GA + \mathrm{d}\lambda \qquad \lambda \in E_k(\mathcal{M}).$$

This leads to

$$\alpha - \varphi^* \omega^k = \pi^* q_0^* GA + \mathrm{d}\pi^* \lambda = \pi^* \mathcal{A}. \tag{3.2}$$

Now, performing a gauge transformation (3.22) on $\mathcal{M}$ with $s = -\lambda$ and calling $\tilde{\varphi}$ the new trivialisation, we have

$$\alpha - \tilde{\varphi}^* \omega^k = \pi^* q_0^* GA = \pi^* \tilde{\mathcal{A}}. \tag{3.3}$$

This is the guiding principle of our construction.

For the sake of simplifying the notation we will write in the following $GM = \bar{\mathcal{M}}$, $GP = \bar{\mathcal{P}}$, $G\pi = \bar{\pi}$, $G\gamma = \bar{\gamma}$ ASO. Let now $M \in C^\infty$-Man and $(P, \pi, M) \in C^\infty$-GPB$^k(M)$. Then by proposition 2.36 $(\bar{\mathcal{P}}, \bar{\pi}, \bar{\mathcal{M}}) \in G^\infty$-GPB$^k(\bar{\mathcal{M}})$. We now define $\mathcal{M} = \bar{\mathcal{M}} \times Q^{0,2}$ and

$$\mathcal{P} = \bar{\mathcal{P}} \times_{E_k(\bar{\mathcal{M}})} \mathcal{P}^k = \{[(\bar{p}, p^k)], (\bar{p}, p^k) \in \bar{\mathcal{P}} \times \mathcal{P}^k | \bar{\pi}(\bar{p}) = q_0 \circ \pi^k(p^k)\} \tag{3.4}$$

where [ ] means equivalence classes with respect to the equivalence relation

$$(\bar{p}, p^k) \sim (\bar{p}', p^{k'}) \Leftrightarrow \bar{p}' = \bar{\gamma}(\bar{p}, \bar{s}), \qquad p^{k'} = \gamma^k(p^k, -q_0^* \bar{s}) \tag{3.5}$$

for some $\bar{s} \in E_k(\bar{\mathcal{M}})$. We call $\nu$ the canonical projection of $\bar{\mathcal{P}} \times \mathcal{P}^k$ onto $\mathcal{P}$. We also define $\pi : \mathcal{P} \to \mathcal{M}$ by

$$\pi([(\bar{p}, p^k)]) = \pi^k(p^k). \tag{3.6}$$

*Theorem 3.7.* The triple $(\mathcal{P}, \pi, \mathcal{M})$ is a generalised principal fibre bundle modelled on $\Lambda^k \mathcal{M}$.

*Proof.* First we define the action $\gamma : \mathcal{P} \times E_k(\mathcal{M}) \to \mathcal{P}$ by†

$$\gamma(p, g) = [(\bar{p}, \gamma^k(p^k, g))] \tag{3.8}$$

where $p = [(\bar{p}, p^k)] \in \mathcal{P}$ and $g \in E_k(\mathcal{M})$. This is well defined, because if we replace $(\bar{p}, p^k)$ by $(\bar{p}', p^{k\prime})$ as in (3.5), then

$$[(\bar{p}, p^{k\prime})] = [(\bar{\gamma}(\bar{p}, \bar{s}), \gamma^k(p^k, g - q_0^* \bar{s}))] = [(\bar{p}, \gamma^k(p^k, g))].$$

Since

$$\pi^k \circ \gamma_g^k = \pi^k, \pi \circ \gamma_g = \pi \quad \text{and} \quad \mathcal{M} = \mathcal{P} \bmod E_k(\mathcal{M}).$$

Let $\phi : P|_U \to P^k|_U$ be a local trivialisation over $U \subset M$; then by proposition 2.36, $\bar{\varphi} = G\phi : \bar{\mathcal{P}}|_{GU} \to \mathcal{P}^k|_{GU}$ is also a local trivialisation; we now define

† This action can be written in the alternative form $\gamma(p, g) = [(\bar{\gamma}(\bar{p}, i^* g), \gamma^k(p^k, g - q_0^* i^* g))]$ where we have defined the canonical injection $i : \bar{\mathcal{M}} \to \mathcal{M}$. The action of $q_0^* i^*$ on a form is to annihilate its components along, and make it independent of, the odd dimensions.

$\varphi:\pi^{-1}(\mathcal{U})\to(\pi^k)^{-1}(\mathcal{U})$ where $u=q_0^{-1}(GU)=GU\times Q^{0,2}$, in the following way: if $p=[(\bar{p},p^k)]$,

$$\varphi(p)=q_0^*\bar{\varphi}(\bar{p})+p^k\in(\pi^k)^{-1}(x) \tag{3.9}$$

which is well defined because $\bar{\pi}(\bar{p})=q_0\circ\pi^k(p^k)=x$ and the sum is in the fibre over $x$. Again we have to check that this is independent of the freedom given by (3.5): writing $p=[(\bar{p}',p^{k'})]$,

$$\begin{aligned}\varphi(p)&=q_0^*\bar{\varphi}(\bar{p}')+p^{k\prime}=q_0^*\bar{\varphi}(\bar{\gamma}(\bar{p},\bar{s}))+\gamma^k(p^k,-q_0^*\bar{s})\\&=q_0^*\bar{\gamma}^k(\bar{\varphi}(\bar{p}),\bar{s})+\gamma^k(p^k,-q_0^*\bar{s})\\&=q_0^*\bar{\varphi}(\bar{p})+q_0^*\bar{s}|_y+p^k-q_0^*\bar{s}|_y=q_0^*\bar{\varphi}(\bar{p})+p^k\end{aligned}$$

where $y=\pi(p)=\pi^k(p^k)$ and we have written $\bar{\gamma}^k$ for the group action on $\Lambda^k\bar{\mathcal{M}}$. Finally, we have to show that $\varphi$ satisfies the compatibility condition with the group action; using (3.8)

$$\begin{aligned}\varphi(\gamma(p,g))&=q_0^*\bar{\varphi}(\bar{p})+\gamma^k(p^k,g)=q_0^*\bar{\varphi}(\bar{p})+p^k+g|_y\\&=\gamma^k(q_0^*\bar{\varphi}(\bar{p})+p^k,g)=\gamma^k(\varphi(p),g).\end{aligned}$$ □

Suppose $a$ is a connection form on $P$; by proposition 2.37, $\bar{\alpha}=Ga$ is a connection form on $\bar{\mathcal{P}}$: we wish now to use $\bar{\alpha}$ to construct a connection form on $\mathcal{P}$. Let us define the projections $\bar{\rho}:\bar{\mathcal{P}}\times\mathcal{P}^k\to\bar{\mathcal{P}}$ and $\rho^k:\bar{\mathcal{P}}\times\mathcal{P}^k\to\mathcal{P}^k$; let $\sigma:\mathcal{P}\to\bar{\mathcal{P}}\times\mathcal{P}^k$ be a section of the map $\nu$, i.e. a choice of a representative couple $(\bar{p},p^k)$ for each element $p\in\mathcal{P}$. Then we may define $\bar{\mu}=\bar{\rho}\circ\sigma:\mathcal{P}\to\bar{\mathcal{P}}$ and $\mu^k=\rho^k\circ\sigma:\mathcal{P}\to\mathcal{P}^k$; a change of section defined by an element of $E_k(\bar{\mathcal{M}})$ as in (3.5) will result in new maps $\bar{\mu}'=\bar{\rho}\circ\sigma'$, $\mu^{k\prime}=\rho^k\circ\sigma'$ which are related to the old ones by

$$\bar{\mu}'=\bar{\gamma}_{\bar{s}}\circ\bar{\mu}\qquad\qquad\mu^{k\prime}=\gamma^k_{-q_0^*\bar{s}}\circ\mu^k. \tag{3.10}$$

The situation is summarised by the following diagram:

$$\begin{tikzcd}
\bar{\mathcal{P}} \arrow[dd, "\bar{\pi}"'] & \bar{\mathcal{P}}\times\mathcal{P}^k \arrow[l, "\bar{\rho}"'] \arrow[r, "\rho^k"] & \mathcal{P}^k \arrow[dd, "\pi^k"] \\
& \mathcal{P} \arrow[u, "\sigma"] \arrow[ul, "\bar{\mu}"] \arrow[ur, "\mu^k"'] \arrow[dr, "\pi"] & \\
\bar{\mathcal{M}} & & \mathcal{M} \arrow[ll, "q_0"]
\end{tikzcd} \tag{3.11}$$

We now define

$$\alpha=\bar{\mu}^*\bar{\alpha}+\mu^{k*}\omega^k \tag{3.12}$$

and we check that it is actually independent of the choice of $\sigma$:

$$\begin{aligned}\bar{\mu}'^*\bar{\alpha}+\mu^{k\prime *}\omega^k&=\bar{\mu}^*\circ\bar{\gamma}_{\bar{s}}^*\bar{\alpha}+\mu^{k*}\circ(\gamma^k_{-q_0^*\bar{s}})^*\omega^k\\&=\bar{\mu}^*(\bar{\alpha}+\bar{\pi}^*\,\mathrm{d}\bar{s})+\mu^{k*}(\omega^k-\pi^{k*}\,\mathrm{d}q_0^*\bar{s})\\&=\bar{\mu}^*\bar{\alpha}+(\bar{\pi}\circ\bar{\mu})^*\,\mathrm{d}\bar{s}+\mu^{k*}\omega^k-(q_0\circ\pi^k\circ\mu^k)^*\,\mathrm{d}\bar{s}\\&=\bar{\mu}^*\bar{\alpha}+\mu^{k*}\omega^k\end{aligned}$$

where we have used relations (2.15), (2.10) and in the last step (3.11).

*Theorem 3.13.* $\alpha$ is a connection form in $\mathscr{P}$.

*Proof.* Let $v \in T_p\mathscr{P}$ be represented by a curve $c: I \to \mathscr{P}$ with $c(0)=p$, $v = \mathrm{d}c/\mathrm{d}t|_{t=0}$. Given a section $\sigma$ as above, we define $\bar{c} = \bar{\mu} \circ c: I \to \bar{\mathscr{P}}$ with $\bar{c}(0) = \bar{\mu}(p) = \bar{p}$ and $c^k = \mu^k \circ c: I \to \mathscr{P}^k$ with $c^k(0) = \mu^k(p) = p^k$. These curves define vectors $\bar{v} = \mathrm{d}\bar{c}/\mathrm{d}t|_{t=0} \in T_{\bar{p}}\bar{\mathscr{P}}$ and $v^k = \mathrm{d}c^k/\mathrm{d}t|_{t=0} \in T_{p^k}\mathscr{P}^k$ clearly

$$\bar{v} = T\bar{\mu}(v) \qquad v^k = T\mu^k(v). \tag{3.14}$$

A different section $\sigma'$ would lead to new vectors $\bar{v}' = T\bar{\mu}'(v)$ and $v^{k\prime} = T\mu^{k\prime}(v)$ and

$$(\bar{v}', v^{k'}) = (T\bar{\gamma}_{\bar{s}}(\bar{v}),\ T\gamma^k_{-q_0^*\bar{s}}(v^k)). \tag{3.15}$$

Since the couples of curves $(\bar{c}, c^k)$ and $(\bar{c}', c^{k'})$ define the same curve $c$, the couples of vectors $(\bar{v}, v^k)$ and $(\bar{v}', v^{k'})$ define the same vector $v$. Thus $v$ can be represented as equivalence classes of couples of vectors under the equivalence relation $(\bar{v}', v^{k'}) \sim (\bar{v}, v^k)$ iff (3.15) holds for some $\bar{s}$. Then (3.12) and (3.14) give

$$i_v\alpha = i_{\bar{v}}\bar{\alpha} + i_{v^k}\omega^k. \tag{3.16}$$

We now choose $\sigma$ in such a way that $\sigma \circ \gamma(p,t) \sim (\bar{p}, \gamma^k(p^k, t))$; if $v = W(\tau)$, the fundamental vector field induced by $\tau \in \mathscr{E}_k(\mathscr{M})$ through the action $\gamma$, then $T\bar{\mu}(W(\tau)) = 0$ and $T\mu^k(W(\tau)) = W^k(\tau)$; thus by (2.11) and (3.11)

$$i_{W(\tau)}\alpha = \mu^{k*}(i_{W^k(\tau)}\omega^k) = \mu^{k*} \circ \pi^{k*}\tau = \pi^*\tau.$$

If $\tau = \mathrm{d}\upsilon$, $\upsilon \in \mathscr{E}_{k-1}(\mathscr{M})$, then

$$\mathscr{L}_{W(\tau)}\alpha = \mu^{k*}\mathscr{L}_{W^k(\tau)}\omega^k = \mu^{k*} \circ \pi^{k*}\,\mathrm{d}\tau = 0. \qquad \square$$

Notice that the proof has been made simple by the choice of section; the reader may convince himself that it is independent of the section e.g. choosing $\sigma$ so that $\sigma \circ \gamma(p,t) = (\bar{\gamma}(\bar{p}, i^*t), \gamma^k(p^k, t - q_0^* i^* t))$ (cf footnote to equation (3.8)).

Having concluded the construction of our bundle with connection, we return to equation (3.1). Since $\mathscr{P}$ is a generalised principal bundle over $\mathscr{M}$ with connection $\alpha$, there exists an $\mathscr{F} \in E_{k+2}(\mathscr{M})$ such that

$$\beta = \mathrm{d}\alpha = \pi^*\mathscr{F}.$$

On the other hand $\mathrm{d}\alpha = \bar{\mu}^*\,\mathrm{d}\bar{\alpha}$ and since $\bar{\alpha}$ is a connection in $\bar{\mathscr{P}}$, there exists $F \in E_{k+2}(M)$ such that

$$\bar{\beta} = \mathrm{d}\bar{\alpha} = \bar{\pi}^* GF.$$

Thus by (3.11)

$$\beta = \bar{\mu}^*\bar{\pi}^* GF = \pi^* \circ q_0^* GF$$

and hence our original condition, (3.1), is satisfied. This implies condition (3.2) and thus there is a gauge (denoted by a tilde) where (3.3) holds, i.e.

$$\tilde{\mathscr{A}} = \mathscr{A} - \mathrm{d}\lambda$$

is completely independent of the odd coordinates. This corresponds to equation (20) of Marchetti and Tonin (1981); the reader may now go back to that work to follow the procedure for constructing the total Lagrangian. We have provided that construction with a rigorous geometrical background based on the extension of Tulczyjew's work to the category of supermanifolds.

We conclude with two remarks. The first is that in the case $k=0$ our construction coincides with that of Bonora *et al* (1981) for an abelian Yang–Mills field. In this case $\mathscr{P}^0 = \mathscr{M} \times Q_0$ and there exists a subgroup of $E_0(\mathscr{M})$, namely the group of constant functions, which is isomorphic to $Q_0$ and acts freely on $\mathscr{P}^0$. Thus we may regard $\mathscr{P}^0$ as a (trivial) principal bundle over $\mathscr{M}$ having as structure group the additive group $Q_0 \in G\mathbb{R}$. Given any generalised principal bundle $\mathscr{P}$ over $\mathscr{M}$ modelled on $\mathscr{P}^0$, the local trivialisations $\varphi : \mathscr{P}|_u \to \mathscr{P}^0|_u$ are just the local trivialisations of $\mathscr{P}$ regarded as a principal $Q_0$-bundle; $\alpha$, which is now a one-form, satisfies the properties of a connection in a principal $Q_0$-bundle. It remains to be shown that $\mathscr{P}$ as defined in (3.4) coincides for $k=0$ with the bundles space of Bonora *et al* (1981c); in fact $\mathscr{P} = \{[(\bar{p}, x, a)],\ (\bar{p}, x, a) \in \bar{\mathscr{P}} \times \mathscr{M} \times Q_0 | \bar{\pi}(\bar{p}) = q_0(x)\}$ and the equivalence relation is (3.5); in each equivalence class we may choose $a=0$ by performing a change of section (in the language of (3.11)) by $\bar{s} \in E_0(\bar{\mathscr{M}})$ defined by $\bar{s}(q_0(x)) = a\ \forall x \in \mathscr{M}$. Thus the effect of the equivalence relation is precisely to cancel the factor $Q_0$ and we remain with $\mathscr{P} = \{(\bar{p}, x) \in \bar{\mathscr{P}} \times \mathscr{M} | \bar{\pi}(\bar{p}) = q_0(x)\}$ which is the definition used in Bonora *et al* (1981c). Notice that this simplification is impossible for $k>0$.

The second remark concerns the difference between the present case and the one of a Yang–Mills theory. There, it was sufficient to take as a fibre of $\mathscr{P}$ over $\mathscr{M}$ the Grassmann enlargement of the fibre of $P$ over $M$; in the present case this is not enough due to the fact that our bundles are soldered to the base manifold (i.e. they consist of forms on $\mathscr{M}$), and thus, enlarging the manifold (from $\bar{\mathscr{M}}$ to $\mathscr{M}$), we must accordingly enlarge the group. Had we not done so, we would have arrived at a geometrically unnatural structure consisting of a bundle space $\mathscr{P}$ modelled on $\Lambda^k\mathscr{M}$ but with a group action isomorphic to $E_k(\bar{\mathscr{M}})$; condition (3.1) would then imply that the most general form of $\mathscr{A}$ is $\mathscr{A} = q_0^* GA + \mathrm{d}q_0^*\bar{\lambda}$ with $\bar{\lambda} \in E_k(\bar{\mathscr{M}})$ and hence the gauge fixing term (1.3) would have been zero. One could then have tried with an $\mathscr{A}$ of the form $\mathscr{A}_{\mu_1 \ldots \mu_{k+1}}(x, \theta, \bar{\theta})\, \mathrm{d}x^{\mu_1} \wedge \ldots \wedge \mathrm{d}x^{\mu_{k+1}}$, i.e. retaining a functional dependence on the odd coordinates but banning components on $\mathrm{d}\theta$ and $\mathrm{d}\bar{\theta}$; this even more hybrid construction would have led to a gauge fixing only for the original gauge freedom (1.2) and the theory would still not be quantisable. We see therefore that the enlargement of the structure group, which is geometrically necessary, leads automatically, in physical terms, to the fixing of all (primary and secondary) gauges.

## Acknowledgments

We are grateful to A Rogers for some fruitful discussion and to M Quiros for kindly making us aware of the recent achievements of the Madrid group.

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